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Nonlinear aspects of the effective theory of dark energy

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Abstract

The goal of this project is to study the effects predicted by scalar-tensor theories of dark energy on small scales. Scalar-tensor theories has been introduce to modify gravity in the infrared. However, large scale modifications of gravity are necessarily associated to short-scale ones. In order to be in accordance with current observational tests on small scales, one can introduce a screening mechanism that is effective in environments where nonlinearities of the scalar field become important, suppressing the extra degrees of freedom. The Vainshtein screening has been already proven to work in many scalar-tensor theories, in particular in the most general second-order covariant theory for gravity coupled to a scalar degree of freedom, the so-called Horndeski theory. The focus of this thesis is on the breaking of this mechanism in the extensions of Horndeski: we retrieve the failure of the screening for beyond Horndeski theories, starting from more general premises. Finally, we tackle the challenge of generalising this result further to DHOST theories.

Chapter 1

Introduction

Scalar-tensor theories for DE and small scale phenomenology

One of the main courses that cosmological research has followed with the purpose of dealing with the accelerated expansion of the universe is represented by modified gravity.
With this term many models of new physics are encompassed, that extend General Relativity in the infrared region and share the goal to give a dynamical alternative to the standard ΛCDM model that would suitably deal with the CC problem.
Since it can be shown that Einstein's theory of gravity is the unique interacting theory of a (graviton) Lorentz invariant helicity-2 massless particle [10][6], it is clear that such models of modified gravity can only proceed all in the same direction: modifying GR by considering additional degrees of freedom. The simplest form that this addition can take is that of a scalar d.o.f. coupled (with even complicated couplings) to the standard 2-d.o.f. tensorial gravity. From here the name given to a wide class of models, which are all together invoked when using the term "scalar-tensor theories".
Finally, General Relativity is a very robust theory, thanks to the considerable number of tests conducted on the astronomical scales and in laboratories. For this reason, any modification to GR should feature some kind of screening mechanism that hinders effects coming from the extension of the theory to arise on small scales (meaning from those that are comparable with the size of our solar system, downwards; the ones on which the great deal of observational tests has been conducted) and thus result in incompatibilities with these stringent constraints.
In this work, we will deal with a wide class of scalar-tensor theories that go under the name of *DHOST* theories and also restrict to a more specific sub-class, called *Horndeski theory* along with its more immediate extension. For these theories, we will study a particular screening mechanism that

arises from terms in the lagrangian containing second-order derivatives that grow in importance in the small scale limit. This type of screening is called *Vainshtein screening*, and is applicable to many ST models[11], the most famous example of which is given by galileon theory[12].
The mechanism has been proven to work also in the immediate generalization to curved space-times of galileons[13], which is the Horndeski theory.
However, one might wonder whether this mechanism will still remain solid while jumping from Horndeski to even more general ST theories.

In chapter I we will give an introduction of the Horndeski theory and its most immediate extension. Further on, we will discuss briefly the issues of Ostrogradski's theorem and the degeneracy as a solution to circumvent its limitations. As a natural consequence, we will introduce the degenerate scalar-tensor theories that we can build from this new perspective, and give a classification inside of this set of models.
In chapter II, we will build the effective theory starting from the quadratic DHOST lagrangian, which will be later restricted to the effective actions used in [1] and to the one relative to class I DHOST theories.
After deriving the equations of motion for both cases, we are ready to analize and discuss the screening mechanism breaking, which is done in chapter III. After a short introduction to the principle of the Vainshtein screening, completed with an example, we produce the proof of the breaking in the case studied by *Kobayashi et alii*.
Finally, we will deal with the equations of motion of the small-scales perturbations around massive sources from the qDHOST effective theory and will show some manipulations in order to approach their solution.

Chapter 2

DHOST theories

For long time the Horndeski theory was thought to be the most general scalar-tensor theory that has the correct number of propagating degrees of freedom (2 tensor modes for the metric sector and a scalar d.o.f.). The requirement of having Euler-Lagrange equations at most of the second order was thought to be crucial to avoid extra ghost-like d.o.f. in the theory and dangerous instabilities, argument supported by the Ostrogradski theorem.
In what follows, we will argue that there exist some immediate and more complex generalizations, that revolve around the key-notion of *degeneracy*.

2.1 Horndeski theory

The Horndeski theory, originally introduced 44 years ago [14], can be thought as the answer to the question: what is the most general theory of a scalar field non-minimally coupled to gravity, featuring equations of motion that are at most of the second order?
In this sense, Horndeski's theory is the most general non-degenerate ST model that respects Ostrogradski's theorem. As such, the theory presents itself in the form of

$$S_H \equiv \int d^4x\sqrt{-g}\sum_{i=2}^{5} L_i^H \tag{2.1}$$

Where the Horndeski lagrangian operators are defined as

$$\begin{cases} L_2^H \equiv G_2(\phi, X) \\ L_3^H \equiv -G_3(\phi, X)\phi^\mu_\mu \\ L_4^H \equiv G_4(\phi, X)R + G_{4X}(\phi, X)[(\phi^\mu_\mu)^2 - (\phi_{\mu\nu})^2] \\ L_5^H \equiv G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} - \frac{1}{6}G_{5X}(\phi, X)[(\phi^\mu_\mu)^3 - 3\phi^\mu_\mu(\phi_{\mu\nu})^2 + 2(\phi_{\mu\nu})^3] \end{cases} \tag{2.2}$$

Here a couple of useful notations are introduced, and are used throughout all these pages. The $\{G_i\}$ are a set of four independent arbitrary functions of the scalar field and its "kinetic term", $X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The derivatives of these functions with respect to their arguments are indicated as $G_{i\phi} \equiv \frac{\partial G_i}{\partial \phi}$ and $G_{iX} \equiv \frac{\partial G_i}{\partial X}$, respectively. Instead, the notation for the covariant derivatives of the scalar field is here $\phi_\mu \equiv \partial_\mu \phi$.

2.2 Beyond Horndeski

The Horndeski theory is a non-degenerate theory, in the sense that will be better precised in the next sections. In [2] the existence of two terms extending the Horndeski theory was proven.

$$\begin{cases} L_4^{bH} \equiv F_4(\phi, X)\epsilon^{\mu\nu\rho}_{\sigma}\epsilon^{\alpha\beta\gamma\sigma}\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma} \\ L_5^{bH} \equiv F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma}\phi_{\sigma\delta} \end{cases} \tag{2.3}$$

These two terms were introduced in this paper as the result of a *disformal transformation* on the quartic and quintic Horndeski lagrangian operators. This is a particular generalization of conformal transformations, were an additional term quadratic in the derivatives of the scalar field is added, along with coefficients that share an arbitrary dependance on X. A generic disformal transformation looks like

$$\tilde{g}_{\mu\nu} \equiv A(\phi, X)g_{\mu\nu} + B(\phi, X)\phi_\mu\phi_\nu \tag{2.4}$$

In fact, these two terms belong to subclasses of degenerate higher order theories that avoid the instability feature thanks to their degeneracy properties (they are respectively terms that fall into specific subsets of quadratic and cubic DHOST theories[1]) In this sense, these terms are a first step towards extending the set of viable scalar-tensor theories beyond the request of having second order e.o.m.
As it will be clear from the detailed analysis in the third chapter, these terms (in particular, the quartic bH one) are responsible for the nonlinear effects arising on small scales and breaking the Vainshtein screening mechanism.

2.3 Ostrogradski instability

Theories described by an action that contain higher order derivatives and a lagrangian that is non-degenerate encounter the so called Ostrogradski instabilities.
These are negative energy solutions to the e.o.m. that propagate inevitably in higher-order non-degenerate theories, as shown in [3].

We will follow briefly the Ostrogradski construction of the hamiltonian

[1]In this work we focused our attention on the first ones.

for a non-degenerate lagrangian of a system describing a point particle, $L[x, x^{(1)}, \cdots, x^{(N)}]$, where $x^{(j)}$ represents j time derivatives acting on the variable. Although the argument here is presentend for finite d.o.f. systems, it can be easily applied also when dealing with fields.
The Euler-Lagrange equations are linear in the $2N$-th derivative:

$$\sum_{i=0}^{N}(-\frac{d}{dt})^i \frac{\partial L}{\partial x^{(i)}} = 0 \tag{2.5}$$

We introduce now the Ostrogradski choice for the canonical variables in the phase space:

$$Q_i \equiv x^{(i-1)} \qquad P_i \equiv \sum_{j=i}^{N}(-\frac{d}{dt})^{j-1} \frac{\partial L}{\partial x^{(j)}} \tag{2.6}$$

The non-degeneracy statement translates into $\frac{\partial^2 L}{\partial x^{(N)} \partial x^{(N)}} \neq 0$, condition that basically allows the existance of a function $\mathcal{A}(Q_1, \cdots, Q_N, P_N)$ such that

$$\frac{\partial L}{\partial x^{(N)}}\Big|_{\substack{x^{(i-1)}=Q_i \\ x^{(N)}=\mathcal{A}}} = P_N \tag{2.7}$$

As a consequence, the following Hamiltonian construction due to Ostrogradski is allowed:

$$\begin{aligned} H &\equiv \sum_{i}^{N} P_i x^{(i)} - L \\ &= P_1 Q_2 + P_2 Q_3 + \cdots + P_{N-1} Q_N + P_N \mathcal{A} - L(Q_1, \cdots, Q_N, \mathcal{A}) \end{aligned} \tag{2.8}$$

with time evolution equations given by:

$$\begin{aligned} \dot{P}_i &\equiv \frac{\partial H}{\partial Q_i} \\ \dot{Q}_i &\equiv -\frac{\partial H}{\partial P_i} \end{aligned} \tag{2.9}$$

From these $2N$ equations, $2(N-1)$ merely restate the (2.6) relations, the other two are equivalent to the relation (2.7) and to the Euler-Lagrange equation.
One immediately sees that the solution of the e.o.m. of this theory generally requires the specification of N initial conditions, hence indicating that we have many d.o.f. propagating.
In addition to that, the Ostrogradski hamiltonian is explicitly unbounded from below with respect to the set of conjugated momenta $\{P_1, \cdots, P_{N-1}\}$, as the dependancy from these fields is linear.
Consequently, unpleasant features appear in our theory, such as negative energy states and vacuum states decaying into groups of positive and negative energy particles.

2.4 Degeneracy

One way to generalize the Horndeski scalar-tensor theory and to overcome the Ostrogradski theorem is to focus on degenerate theories, that feature a non-invertible kinetic lagrangian. Generally, a dynamical system is thus allowed to have higher order derivatives present at the lagrangian level and higher order time derivatives at the e.o.m. level. Nonetheless, the degeneracy of the kinetic lagrangian implies the existence of additional constraints on the phase space that reduce the actual number of propagating degrees of freedom.

In the case of scalar-tensor theories, one can follow a thorough hamiltonian analysis in order to properly analyze the number of degrees of freedom and study the consequences of a degenerate lagrangian on it. We will first review here in brief the analysis as conducted in [4] on a simplified toy model, in order to show how degeneracy can circumvent Ostrogradski theorem and avoid the instabilities predicted by it.

Let's take a point particle, $\phi(t)$, who's dynamics is regulated by terms with higher number of time derivatives, coupled to n regular degrees of freedom, $q^i(t)$, $i = 1, \cdots, n$, who feature only standard first time derivative lagrangian terms. The lagrangian describing this toy model is thus taken to be:

$$L = \frac{a}{2}\ddot{\phi}^2 + \frac{k_0}{2}\dot{\phi}^2 + \frac{1}{2}k_{ij}\dot{q}^i\dot{q}^j + b_i\ddot{\phi}\dot{q}^i + c_i\dot{\phi}\dot{q}^i - V(\phi, q) \tag{2.10}$$

The equations of motion, obtained by variation of (2.10) with respect to, respectively, $\phi(t)$ and $q^i(t)$ are:

$$\begin{aligned} a\ddddot{\phi} - k_0\ddot{\phi} + b_i\dddot{q}^i - c_i\ddot{q}^i - \frac{\partial V}{\partial \phi} = 0 \\ k_{ij}\ddot{q}^i + b_j\dddot{\phi} + c_j\ddot{\phi} + \frac{\partial V}{\partial q^j} = 0 \end{aligned} \tag{2.11}$$

One can immediately guess from this higher order equations at this stage that this toy model features extra ghost-like d.o.f. propagating, since the solution of (2.11) generally requires the specification of more than two initial conditions.

To conduct the analysis we first should replace the higher order time derivatives by introducing a new variable, Q, and associate it to $\dot{\phi}$ with an additional constraint in the lagrangian:

$$L = \frac{a}{2}\dot{Q}^2 + \frac{k_0}{2}Q^2 + \frac{1}{2}k_{ij}\dot{q}^i\dot{q}^j + b_i\dot{Q}\dot{q}^i + c_iQ\dot{q}^i - V(\phi, q) - \lambda(Q - \dot{\phi}) \tag{2.12}$$

The equation of motion for the theory reformulated in this fashion are:

$$\begin{aligned}&a\ddot{Q} + b_i\ddot{q}^i = k_0 Q + c_i\dot{q}^i - \lambda\\&b_j\ddot{Q} + k_{ij}\ddot{q}^i = -c_j\dot{Q} - V_j\\&Q = \dot{\phi}\\&\dot{\lambda} = -V_\phi\end{aligned} \tag{2.13}$$

Where the notations $\frac{\partial V}{\partial \phi} \equiv V_\phi$ and $\frac{\partial V}{\partial q^i} \equiv V_i$ were used for simplicity. It is immediate to check that the two systems of equations (2.11) and (2.13) are equivalent.
Defining now the kinetic matrix as the one built up with the coefficients of the terms in (2.12) that are quadratic in the time derivatives, we can write it explicitly as:

$$M \equiv \begin{pmatrix} a & b_i \\ b_j & k_{ij} \end{pmatrix} \tag{2.14}$$

This matrix is the key to the presence or not of instabilities in this model. Basically, when it is invertible, the first two equations of (2.13) can be solved to express $\ddot{Q}$ and $\ddot{q}^i$ in terms of the first order time derivatives. This way, it is clear that the solution of (2.13) requires $2(n+2)$ initial conditions to be specified for the variables $\{q^i, \dot{q}^i, Q, \dot{Q}, \lambda, \phi\}$. Thus, there are $(n+2)$ degrees of freedom propagating in this system, including the extra ghost-like d.o.f. that is the Ostrogradski instability, with consequences as seen in the previous section.

Degeneracy of (2.14) is the key for introducing those constraints on the phase space that will cancel the ghost.
We require the degeneracy to be restricted to the ϕ-sector and its couplings to the n ordinary d.o.f.: the k_{ij} matrix is assumed to be invertible. Hence, the degeneracy condition translates into:

$$\begin{aligned}&0 = detM = det(k)(a - b_i b_j (k^{-1})^{ij})\\&a - b_i b_j (k^{-1})^{ij} = 0\end{aligned} \tag{2.15}$$

From this condition, a null eigenvector for the kinetic matrix can be singled out:

$$v = \begin{pmatrix} v^0 \\ v^i \end{pmatrix} = \begin{pmatrix} -1 \\ (k^{-1})^{ij} b_j \end{pmatrix} \tag{2.16}$$

which generates the one-dimensional kernel of (2.14). The system of equations (2.13) can be projected along v, operation that will eventually result in an equivalent system but with reduced number of d.o.f.; in particular, we can introduce the new variable $x^i = q^i + v^i Q$ that will replace the q^i

variable, and obtain the equivalent system:

$$
\begin{aligned}
&c_i\dot{x}^i + k_0 Q + v^i V_i = \lambda \\
&k_{ij}\ddot{x}^i + c_j\dot{Q} + V_j = 0 \\
&Q = \dot{\phi} \\
&\dot{\lambda} = -V_\phi
\end{aligned}
\tag{2.17}
$$

After replacing the third and fourth of these equations back in the first two, and taking the derivative of the first one, one gets the result:

$$
\begin{aligned}
&(k_0 - v^i v^j V_{ij})\ddot{\phi} + c_i\ddot{x}^i = -v^i V_{ij}\dot{x}^j - v^i V_{i\phi}\dot{\phi} - V_\phi \\
&c_j\ddot{\phi} + k_{ij}\ddot{x}^i = -V_j
\end{aligned}
\tag{2.18}
$$

In this new system, equivalent to the original degenerate one, we see that the kinetic matrix looks like

$$
\tilde{M} \equiv \begin{pmatrix} (k_0 - v^i v^j V_{ij}) & c_i \\ c_j & k_{ij} \end{pmatrix}
\tag{2.19}
$$

and we can recognize right away that the variables are now $\{\phi, x^i\}$, thus the initial conditions to specify are now $2(n+1)$ and there are $(n+1)$ d.o.f. propagating in the degenerate version of the toy model, just as desired.
This conclusion in valid if we assume the new kinetic matrix (2.19) to be non-degenerate. The degeneracy can be extended further, by taking $\tilde{M}$ to be degenerate and repeating the same steps, one can reduce even more the number of degrees of freedom.

As for the hamiltonian analysis, one first needs to introduce conjugated momenta associated with the configuration variables

$$
\begin{aligned}
&\begin{pmatrix} P \\ p_i \end{pmatrix} \equiv \begin{pmatrix} \frac{\partial L}{\partial \dot{Q}} \\ \frac{\partial L}{\partial \dot{q}^i} \end{pmatrix} = M \begin{pmatrix} \dot{Q} \\ \dot{q}^i \end{pmatrix} + \begin{pmatrix} 0 \\ c_i Q \end{pmatrix} \\
&\pi_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \lambda
\end{aligned}
\tag{2.20}
$$

and then construct the Hamiltonian through a Legendre transformation of the (2.10) lagrangian

$$
H = P\dot{Q} + p_i\dot{q}^i + \pi_\phi\dot{\phi} - L
\tag{2.21}
$$

resulting in an hamiltonian system describing the dynamics of the $2(n+2)$ canonical variables $\{P, q_i, \pi_\phi\}$ and $\{Q, q^i, \phi\}$ or, in other words, the $(n+2)$ degrees of freedom, in the non-degenerate case.

The final step left to complete is to construct the two constraints that will

eventually effectively cancel out 2 of the canonical variables.
The first constraint can be extracted from the existence of the non trivial kernel of the kinetic matrix, exploiting the same null eigenvalue defined in the previous lagrangian analysis:

$$\Omega \equiv \begin{pmatrix} v^0 & v^i \end{pmatrix} \cdot \begin{pmatrix} P \\ p_i - c_i Q \end{pmatrix} = v^i(p_i - c_i Q) - P = 0 \tag{2.22}$$

as a consequence of projecting (2.20) along the eigenvector. Evidently, this constraint allows express P as a function of the variables (p_i, Q) and doing so eliminate one of the two variables.
The second constraint, arises from the request of the Ω constraint being constant in time. In other words, we want the time evolution defined by the "projected" Hamiltonian function[2]

$$H_T = \frac{1}{2}(k^{-1})^{ij}(p_i - c_i Q)(p_j - c_j Q) - \frac{k_0}{2}Q^2 + V(\phi, Q) + \pi_\phi Q + \mu\Omega \tag{2.23}$$

to leave Ω invariant:

$$\Psi \equiv \dot{\Omega} = \{\Omega, H_t\} = c_i(k^{-1})^{ij}(p_j - Qc_j) + k_0 Q + v^i V_i - \pi_\phi = 0 \tag{2.24}$$

Clearly, with this constraint one can replace π_ϕ with a function of other canonical variables, canceling out an additional momenta from the count of independent variables in the system, leaving only $(n+1)$ degrees of freedom.
One could go on with this procedure, looking for a third constraint. The candidate would be

$$\{\Omega, \Psi\} = \Delta \tag{2.25}$$

where Δ is defined as $det\tilde{M} = det(k)\Delta$. As a consequence, as long as the determinant of the kinetic matrix isn't furtherly degenerate, additional constraints reducing the phase space dimension are not given.

In this work, the subject is focused on the quadratic DHOST theories. For this reason, the following lines will be dedicated to the construction of the quadratic DHOST lagrangian only, leaving out of this pages the cubic DHOST theory.

2.5 Quadratic DHOST

We start here with an action composed by a term that is basically a generalization of Einstein-Hilbert action, and a second term that depends quadratically on the second derivatives of the field, $\phi_{\mu\nu} \equiv \partial_\mu \partial_\nu \phi$:

$$S = \int d^4x \sqrt{-g} f(\phi, X) R + \int d^4x \sqrt{-g} C^{\mu\nu,\rho\sigma} \phi_{\mu\nu}\phi_{\rho\sigma} \tag{2.26}$$

[2] H_T features a lagrange multiplier,μ, that encodes the first constraint in the dynamics of the system.

where here the tensor $C^{\mu\nu,\rho\sigma}$ is a function of only ϕ and ϕ_μ. By imposing the natural symmetries

$$C^{\mu\nu,\rho\sigma} = C^{\nu\mu,\rho\sigma} = C^{\mu\nu,\sigma\rho} = C^{\rho\sigma,\mu\nu}$$

we are able to obtain the general structure of the C tensor:

$$\begin{aligned} C^{\mu\nu,\rho\sigma} =& \frac{1}{2}a_1(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) + a_2 g^{\mu\nu}g^{\rho\sigma} + \frac{1}{2}a_3(\phi^\mu\phi^\nu g^{\rho\sigma} + \phi^\rho\phi^\sigma g^{\mu\nu}) \\ &+ \frac{1}{4}a_4(\phi^\mu\phi^\rho g^{\nu\sigma} + \phi^\nu\phi^\sigma g^{\mu\rho} + \phi^\mu\phi^\sigma g^{\nu\rho} + \phi^\nu\phi^\rho g^{\mu\sigma}) + a_5\phi^\mu\phi^\nu\phi^\rho\phi^\sigma \end{aligned} \tag{2.27}$$

Substitution of (2.27) in the action (2.26) brings us as a result the quadratic DHOST action, in terms of the 5 DHOST lagrangian operators, L_I,

$$S_{DHOST} = \int d^4x\sqrt{-g}[f(\phi, X)R + \sum_I a_I(\phi, X)L_I] \tag{2.28}$$

$$\begin{array}{ccc} L_1 \equiv \phi^{\mu\nu}\phi_{\mu\nu} & L_2 \equiv (\phi^\mu_\mu)^2 & L_3 \equiv \phi^\sigma_\sigma\phi^\mu\phi_{\mu\nu}\phi^\nu \\ L_4 \equiv \phi^\mu\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho & L_5 \equiv (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2 & \end{array} \tag{2.29}$$

In fact, other terms could be added to (2.28), such as terms that are at most linear in the second derivatives of the field

$$\int d^4x\sqrt{-g}L_{other} = \int d^4x\sqrt{-g}[P(\phi, X) + Q_1(\phi, X)g^{\mu\nu}\phi_{\mu\nu} + Q_2(\phi, X)\phi^\mu\phi_{\mu\nu}\phi^\nu] \tag{2.30}$$

For the sake of the degeneracy analysis, these terms are neglegible as they do not interfere with the kinetic matrix. For this reason, we will forget about them when deriving the degeneracy conditions for quadratic DHOST theories. Nonetheless, we will actually take into consideration the shift symmetric one of these terms in the study of the effective theory, later on, as it have a certain contribution to it.

2.6 Degeneracy in scalar-tensor theories

In order to study the possible degeneracy of the (2.28) lagrangian we need to reformulate it, analogously to what we did in the toy model in the previous chapter.

As in [4] we shall introduce $A_\mu \equiv \phi_\mu$ and replace the new variable in the lagrangian of the theory, adding also a lagrange multiplier to enforce its relation to the original scalar field. The reformulated lagrangian should look like

$$S[g_{\mu\nu}, \phi, A_\mu, \lambda^\mu] = \int d^4x\sqrt{-g}[fR + C^{\mu\nu,\rho\sigma}\nabla_\mu A_\nu\nabla_\rho A_\sigma + \lambda^\mu(A_\mu - \phi_\mu)] \tag{2.31}$$

Second of all, since the degeneracy study involves time derivatives we need to "dig them out" of the covariant formalism: we shall conduct a covariant ADM decomposition on our lagrangian, in order to separate time-like and space-like components of the tensors in our theory.
Assuming the existence of a slicing of the spacetime into 3-dimensional space-like hypersurfaces and a normal time-like direction, we can introduce the time-like unit vector n^a, normal to the hyper-surfaces, such that $n^a n_a = -1$.
This slicing allows us to introduce the metric induced on the 3-D hypersurfaces, which we can also use to project tensors on the hypersurfaces themselves, and thus recover the tensor's space-like component:

$$h_{ab} \equiv g_{ab} + n_a n_b \tag{2.32}$$

The decomposition of A_μ into its spatial and normal projections is thus given by

$$\begin{cases} \hat{A}_a = h_a^b A_b \\ A^* = n^a A_a \end{cases} \tag{2.33}$$

The next step is to introduce the time direction $t^a \equiv \frac{\partial}{\partial t}$, in order to define the lapse function, N, and the shift vector, N^a

$$t^a \equiv N n^a + N^a \tag{2.34}$$

Now we can define the time derivative of the projected tensors, for example

$$\begin{cases} \dot{A}^* \equiv t^a \nabla_a A^* \\ \dot{\hat{A}} \equiv h_a^b \mathcal{L}_t \hat{A}_b = h_a^b (t^c \nabla_c \hat{A}_b + \hat{A}_c \nabla_b t^c) \end{cases} \tag{2.35}$$

The key now is the decomposition of the derivative ∇_μ, which will allow us to extract the terms quadratic in the time derivatives and compose the elements of the kinetic matrix for this theory.
Such a decomposition would look like this

$$\nabla_a A_b \equiv D_a \hat{A}_b - A^* K_{ab} + n_{(a}(K_{b)c}\hat{A}^c - D_{b)}A^*) + \frac{1}{N} n_a n_b (\dot{A}^* - N^c D_c A^* - N\hat{A}_c a^c) \tag{2.36}$$

Here we used D_a, the covariant derivative induced on the hypersurfaces, and a^c, the acceleration defined as $a^c \equiv n^b \nabla_b n^c$. We also made use of the extrinsic curvature tensor, defined as

$$K_{ab} \equiv \frac{1}{2N}(\dot{h}_{ab} - D_{(a}N_{b)}) \tag{2.37}$$

The only time derivatives appear to be $\dot{A}^*$ and the derivative of the induced metric elements, present in K_{ab}. The relevant part of the derivative term we just decomposed, hence appear to be

$$\nabla_a A_b \Big|_{kinetic} = \lambda_{ab} \dot{A}^* + \Lambda_{ab}^{cd} K_{cd} \tag{2.38}$$

where, for brevity, $\lambda_{ab} \equiv \frac{n_a n_b}{N}$ and $\Lambda_{ab}^{cd} \equiv -A^* h^c_{(a} h^d_{b)} + 2n_{(a} h^{(c}_{b)} \hat{A}^{d)}$.
Eventually, the kinetic part of the lagrangian results in

$$\mathcal{L}_{kin} = C^{ab,cd}\lambda_{ab}\lambda_{cd}A^{*2} + 2C^{ab,cd}\lambda_{ab}\Lambda^{ef}_{cd}A^* K_{ef} + C^{ab,cd}\Lambda^{ef}_{ab}\Lambda^{gh}_{cd}K_{ef}K_{gh} \tag{2.39}$$

This is the result for what concerns the scalar sector. To complete the calculation of the kinetic matrix, though, we need to evaluate also the contribution coming from the gravitational term of the action, the one containing the Ricci scalar.
To do this, we need the Gauss-Codazzi equation

$$R = {}^{(3)}R + K_{\mu\nu}K^{\mu\nu} - K^2 - 2\nabla_\mu(a^\mu - Kn^\mu) \tag{2.40}$$

Substituting this in the original action, we get

$$S_g = \int d^4x\sqrt{-g}\{f[K_{\mu\nu}K^{\mu\nu} - K^2 + {}^{(3)}R] + 2\nabla_\mu(a^\mu - Kn^\mu)\} \tag{2.41}$$

From this part of the action, two contributions to the kinetic emerge: one to the mixed term

$$\mathcal{B}^{ab}_{grav} \equiv 2f_X\frac{A^*}{N}h_{ab}\dot{A}^* K^{ab} \tag{2.42}$$

and one to the term quadratic in K^{ab}

$$\mathcal{K}^{ab,cd}_{grav} \equiv fh^{a(c}h^{d)b} - fh^{ab}h^{cd} + 2f_X(\hat{A}^a\hat{A}^b h^{cd} + \hat{A}^c\hat{A}^d h^{ab}) \tag{2.43}$$

Together with the contribution coming from the scalar sector, that is

$$\begin{aligned} \mathcal{A} &\equiv C^{ab,cd}\lambda_{ab}\lambda_{cd} \\ \mathcal{B}^{ef}_\phi &\equiv C^{ab,cd}\Lambda^{ef}_{ab}\lambda_{cd} \\ \mathcal{K}^{ef,gh}_\phi &\equiv C^{ab,cd}\Lambda^{ef}_{ab}\Lambda^{gh}_{cd} \end{aligned} \tag{2.44}$$

so that the kinetic matrix is finally given by

$$\mathcal{M} \equiv \begin{pmatrix} \mathcal{A} & \mathcal{B}^{cd}_\phi + \mathcal{B}^{cd}_{grav} \\ \mathcal{B}^{ab}_\phi + \mathcal{B}^{ab}_{grav} & \mathcal{K}^{ab,cd}_\phi + \mathcal{K}^{ab,cd}_{grav} \end{pmatrix} \tag{2.45}$$

The degeneracy condition wants the determinant of this matrix to be zero, and imposing such a condition on (2.45) leads to the expression

$$D_0(X) + D_1(X)A^{*2} + D_2(X)A^{*4} = 0 \tag{2.46}$$

Evidently, the three terms must vanish indipendently, resulting in the set of three degeneracy conditions [3]

$$0 = D_0(X) \equiv 4(a_1 + a_2)[2Xf(2a_1 - 2Xa_4 - 2f_X) + 2f^2 + 8X^2 f_X^2] \tag{2.47}$$

[3] In the literature about this topic, one can find these three degeneracy conditions expressed in an alternative fashion, due to the different definition of X.

$$0 = D_1(X) \equiv 4[X^2 a_1(a_1 + 3a_2) - 2f^2 + 8Xfa_2]a_4 + 16X^2 f(a_1 + a_2)a_5 - 16Xa_1^3 \\ - 4(f + 4Xf_X + 12Xa_2)a_1^2 - 16(f + 5Xf_X)a_1 a_2 - 8X(3f - 4Xf_X)a_1 a_3 \\ - X^2 f a_3^2 - 16f_X(f + 2Xf_X)a_2 + 8ff_X a_1 - 8f(f - Xf_X)a_3 + 12ff_X^2 \tag{2.48}$$

$$0 = D_2(X) \equiv 4[2f^2 - 8Xfa_2 - 4X^2 a_1(a_1 + 3a_2)]a_5 + 4a_1^3 + 4(2a_2 + 2Xa_3 + 2f_X)a_1^2 \\ + 12X^2 a_1 a_3^2 + 8Xfa_3^2 + 8(f + Xf_X)a_1 a_3 + 16f_X a_1 a_2 + 4f_X^2 a_1 + 8f_X^2 a_2 + 8ff_X a_3 \tag{2.49}$$

2.7 DHOST classification

The quadratic DHOST theories can be classified based on these three degeneracy conditions, as seen in [5].
Up to now, the entire quadratic DHOST space of theories is spanned by 6 indipendent functions of the scalar field and its kinetic term. From the three degeneracy conditions, (2.47)-(2.49), one can obtain a classification of the quadratic DHOST theories based on some fixed relations between the coefficients $\{a_i\}$ that reduce the number of independent functions. Actually, based on the first degeneracy condition, one can immediately distinguish three main classes. Indeed, the $D_0(X)$ function vanishes if one of the following cases is true

$$\begin{cases} a_1 + a_2 = 0 & (I) \\ 2Xf(2a_1 - 2Xa_4 - 2f_X) + 2f^2 + 8X^2 f_X^2 = 0 & (II) \\ f = 0 & (III) \end{cases} \tag{2.50}$$

2.7.1 Class I

In the first case, the *class (I)* is defined by the relation

$$a_1 = -a_2 \tag{2.51}$$

The degeneracy conditions (2.48) and (2.49) can be used to express a_4 and a_5 coefficients in terms of the others. Two sub-cases arise now.

In the first, by assuming $f - 2Xa_2 \neq 0$, we can express this sub-class, that we should call *class Ia*, through the relations

$$a_4 = \frac{1}{2(f - 2Xa_2)^2}[-8Xa_2^3 + (3f + 16Xf_X)a_2^2 + (-8X^2 f_X + 6Xf)a_3 a_2 \\ - X^2 f a_3^2 - 2f_X(3f + 4Xf_X)a_2 + 2f(Xf_X - f)a_3 + 3ff_X^2] \tag{2.52}$$

$$a_5 = \frac{1}{2(f - 2Xa_2)^2}(a_2 - Xa_3 - f_X)(f_X a_2 + 2fa_3 - a_2^2 - 3Xa_2 a_3) \tag{2.53}$$

The theories in this class are defined by means of three independent functions $\{a_2, a_3, f\}$.

In the second sub-class, *class Ib*, we assume instead that $f = 2Xa_2$, so that

$$a_1 = -a_2 = -\frac{f}{2X} \tag{2.54}$$

$$a_3 = \frac{1}{2X}(f - 2Xf_X) \tag{2.55}$$

In class Ib, however, dwell theories that are degenerate in the metric sector, thus not interesting for us that require degeneracy in the ϕ-sector or in the scalar-tensor coupling.

2.7.2 Class II

Class II, characterized by

$$2Xf(2a_1 - 2Xa_4 - 2f_X) + 2f^2 + 8X^2 f_X^2 = 0 \tag{2.56}$$

also features two sub-classes. As before, we can express a_4 and a_5 in terms of the other coefficients. By plugging the result in (2.56), one can get

$$(f + 2Xa_1)[(4f^2 - 2Xf(8a_2 + 2a_1 - 2Xa_3 + 2f_X) + 8X^2 f_X(a_1 + 3a_2)] = 0 \tag{2.57}$$

By assuming $f + 2Xa_1 \neq 0$, one can define class IIa as

$$a_3 = \frac{1}{4X^2 f}[-4f(f - Xf_X) + 4X(f - 2Xf_X)a_1 - 8X(-2f + 3Xf_X)] \tag{2.58}$$

$$a_4 = \frac{1}{2X^2 f}[f^2 - 2fXf_X + 4X^2 f_X^2 + 2Xfa_1] \tag{2.59}$$

$$a_5 = \frac{1}{4f^2X^3}[4f(f^2 - 3fXf_X + 2X^2 f_X^2) + (-6Xf^2 + 16X^2 ff_X - 12X^3 f_X^2)a_1 - 4X(2f - 3Xf_X)^2 a_2] \tag{2.60}$$

The three arbitrary functions spanning the space of theories in class IIa are $\{f, a_1, a_2\}$.

The second sub-class, is defined by the condition $f = -2Xa_1$. This class is very similar to class Ib, in the sense that here also the metric sector is the one hiding the degeneracy. This class is defined by the relations

$$a_1 = -\frac{f}{2X} \tag{2.61}$$

$$a_4 = 2f_X\left(\frac{f_X}{f} - \frac{1}{2X}\right) \tag{2.62}$$

$$a_5 = \frac{1}{-32X^3 f(f-2Xa_2)}[-16X(4Xf_X f - f^2 - 4X^2 f_X^2)a_2 - 2Xf(-16X^2 f_X - 8X^3 a_3 - 4f)a_3 + 4(Xf_X f^2 - 2X^3 f_X^3 + 2X^2 f_X^2 f - f^3)] \tag{2.63}$$

2.7.3 Class III

The last class is characterized by $f = 0$. From this condition,

$$D_1(X) = 0 \qquad D_2(X) = 0 \tag{2.64}$$

can be exploited to obtain

$$a_4 = \frac{a_1}{X} \quad a_5 = \frac{a_1^2 + 2a_1a_2 + 2a_1a_3X + 3a_3^2X^2}{4X^2(a_1+3a_2)} \tag{2.65}$$

as these relations describe the sub-class IIIa, as long as $a_1 \neq -3a_2$. This subset of DHOST theories is spanned by the three independent functions $\{a_1, a_2, a_3\}$. Furthermore, the intersection $Ia \cap IIIa$ is not only a non-trivial one, but sub-class of particular interest: being spanned by only two arbitrary coefficients, it contains the quartic beyong Horndeski lagrangian, L_4^{bH}.

The case in which $a_1 = -3a_2$ instead, characterizes the class IIIb through the set of relations:

$$a_1 = -3a_2 \quad a_2 = Xa_3 \quad f = 0 \tag{2.66}$$

Finally, the third sub-class, IIIc, specified by

$$f = 0 \quad a_1 = 0 \tag{2.67}$$

And is spanned by four independent functions of (ϕ, X).

2.8 $\mathcal{L}_4^{\mathbf{H}} + \mathcal{L}_4^{\mathbf{bH}}$ as quadratic DHOST

The quartic lagrangian operators of the Horndeski theory

$$L_4^H = G_4(\phi, X)R + G_{4X}(\phi, X)[(\phi_\mu^\mu)^2 - (\phi_{\mu\nu})^2] \tag{2.68}$$

can be retrieved from the general quadratic DHOST lagrangian as a particular case, corresponding to the choice of coefficients:

$$f = G_4 \quad a_1 = -a_2 = -G_{4X} \quad a_3 = a_4 = a_5 = 0 \tag{2.69}$$

From these relations it is clear that this term belongs to the class Ia.
The quartic beyond Horndeski lagrangian, instead,

$$L_4^{bH} = F_4(\phi, X)\epsilon_\sigma^{\alpha\beta\gamma}\epsilon^{\mu\nu\rho\sigma}\phi_\alpha\phi_\mu\phi_{\beta\nu}\phi_{\gamma\rho} \tag{2.70}$$

appears as the choice of coefficients

$$f = 0 \quad a_1 = -a_2 = -2XF_4 \quad a_3 = -a_4 = 2F_4 \quad a_5 = 0 \tag{2.71}$$

This set of relations select too a subclass of class Ia. In particular, we saw in the previous section that L_4^{bH} belongs to the intersection with class IIIa.

But in particular, a more interesting class of theories is represented by the one featuring degeneracy due to the scalar sector alone.
By this definition, this class, that we shall call from now on *class A*, is associated to theories that satisfy the condition

$$0 = \mathcal{A} = (a_1 + a_2) + (a_3 + a_4)A^{*2} + a_5 A^{*4} \tag{2.72}$$

The three degeneracy conditions to which (2.72) is equivalent, select a peculiar subset of class Ia (as one can read out from $a_1 = -a_2$) characterized by

$$a_5 = 0 \quad \Longrightarrow \quad f_X - a_2 + Xa_3 = 0 \tag{2.73}$$

This condition reduces the number of independent functions to two, $\{f, a_2\}$, and allows the indentification of class A with the $L_4^H + L_4^{bH}$ theory, upon the choice of the combination of coefficients

$$f = G_4 \quad a_1 = -a_2 = -G_{4X} - 2XF_4 \quad a_3 = -a_4 = 2F_4 \tag{2.74}$$

This identification supports the statement that the combination of quartic Horndeski and beyond lagrangians is the most general quadratic ST theory that is degenerate in the ϕ-sector alone, and thus remains degenerate even when the metric is non-dynamical (that is, even in the case of a flat space-time).

Chapter 3

Effective theory of dark energy

3.1 Perturbations in quasi-static approximation

We start from the lagrangian of a quadratic DHOST scalar-tensor theory, plus the matter lagrangian featuring a minimal coupling to the metric,

$$S = S_g^{DHOST}[g_{\mu\nu}, \phi] + S_{mat}[g_{\mu\nu}, \psi_m, A_\mu, \cdots] \tag{3.1}$$

that describes non-relativistic matter sources that are composed of standard model particles and cold dark matter. That means that the stress-energy tensor associated to the matter component is that of a perfect preassureless fluid, $T^{\mu\nu} = diag(-\rho_m, 0, 0, 0)$.
In details, the gravitational part of the action is taken to have the form of a generic quadratic DHOST theory of a scalar degree of freedom non-minimally coupled to gravity, plus an arbitrary function of the scalar field and its kinetic term (the shift symmetric term among the ones that in [4] go under the name of "others"):

$$S_g^{DHOST}[g_{\mu\nu}, \phi] = \int d^4x\sqrt{-g}[F(\phi, X)R + a_I(\phi, X)L_I + P(\phi, X)] \tag{3.2}$$

with the lagrangian operators L_I represented explicitly by:

$$\begin{gathered} L_1 = \phi_{\mu\nu}\phi^{\mu\nu} \quad L_2 = (\phi^\mu_\mu)^2 \quad L_3 = (\phi^\mu_\mu)(\phi^\rho\phi_{\rho\sigma}\phi^\sigma) \\ L_4 = \phi^\mu\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho \quad L_5 = (\phi^\rho\phi_{\rho\sigma}\phi^\sigma)^2 \end{gathered} \tag{3.3}$$

Here we remind the reader that we make use of the notation in which $X = -\frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu$ and the greek indices on scalar fields stand for covariant derivatives acting on the fields themselves, $\phi_{\mu\nu} = \nabla_\mu\partial_\nu\phi$.

From this starting point we want to build the effective theory, valid for small

scales, for the small fluctuations of the metric potentials and of the scalar field having the matter overdensity, δ, as a source.
To do so, we proceed by perturbing the metric and the scalar field around a flat Friedman-Robertson-Walker background, expanding the (3.1) action in terms of the perturbations $h_{\mu\nu}$, π and δ following

$$g_{\mu\nu} \to g_{\mu\nu}(t) + \frac{h_{\mu\nu}(t,\vec{x})}{\widetilde{M}_{Pl}}, \quad h_{00} = -2\Phi(t,\vec{x}) \quad h_{ij} = -2a^2(t)\Psi(t,\vec{x})\delta_{ij} \tag{3.4}$$

$$\phi \to \phi(t) + \pi(t,\vec{x}) \tag{3.5}$$

$$\rho \to \rho(t)[1+\delta(t,\vec{x})] \tag{3.6}$$

where $\widetilde{M}_{Pl}$ is a mass scalle of order of the actual Planck mass.
The expansion of the lagrangian in (3.1) has been done in the Newtonian gauge, in order to keep only the scalar perturbations of the metric.
Moreover, this procedure was conducted under the *quasi-static approximation*, as the galaxies and halos, that constitute the matter sources around which our analysis is thought to be conducted, are assumed to be fixed in the same configuration; similarly, also the scalar field's profile is assumed to be basically frozen in time: it's dynamics, around small scales, is thought to be dominated by spatial derivatives. This idea translates in terms of lagrangian operators as

$$(\nabla\epsilon)^2 \gg (\partial_t\epsilon)^2 \tag{3.7}$$

where ϵ stands for the any of the fields π,Φ or Ψ. Furthermore, we want to select only those terms in the equation of motion that present at least two spatial derivatives per field, because they grow in importance as we go to smaller scales: these are generated by terms in the effective lagrangian that contain a number of derivatives, $\#(\nabla)$, and a number of fields, $\#(\epsilon)$, that satisfy

$$\#(\nabla) \geq 2(\#(\epsilon)-1)$$

and are the operators that will significally contribute to the mechanism of screening.
All these steps were conducted by first splitting the metric and the resulting perturbations in an ADM-like fashion into components respectively parallel and orthogonal to space-like hypersurfaces. These calculations were conducted with the help of the Mathematica package *xPand*. [7]

This said, the results for every quadratic DHOST lagrangian operator are listed below, organized in quadratic and non-linear part:

3.1.1 $\sqrt{-g}F[\phi,X]R$

$$\mathcal{L}^{(NL)}_{R,Eff} = \frac{1}{a}[-2F_X(\nabla_i\Phi)(\nabla_j\pi)[\Pi^{ij}] + 4F_X(\nabla_i\Psi)(\nabla_j\pi)[\Pi^{ij}]] \tag{3.8}$$

$$\mathcal{L}^{(2)}_{R,Eff} = a[(4XF_X)\Phi\nabla^2\Phi + (-2F)\Psi\nabla^2\Psi + (2H\dot{\phi}F_X + 2\ddot{\phi}F_X + 4X\ddot{\phi}F_{XX} - 2F_\phi + 4XF_{\phi X})\Phi\nabla^2\pi + (-4H\dot{\phi}F_X - 4\ddot{\phi}F_X - 8X\ddot{\phi}F_{XX} + 4F_\phi - 8XF_{\phi X})\Psi\nabla^2\pi + (4F - 8XF_X)\Psi\nabla^2\Phi + (2\dot{\phi}F_X)\dot{\Phi}\nabla^2\pi + (-4\dot{\phi}F_X)\dot{\Psi}\nabla^2\pi + (6H^2F_X + 3\dot{H}F_X)\pi\nabla^2\pi] \quad (3.9)$$

3.1.2 $\sqrt{-g}a_1[\phi, X]L_1$

$$\mathcal{L}^{(NL)}_{1,Eff} = \frac{1}{a^3}[(-\frac{a_{1X}}{2})(\nabla\pi)^2[\Pi^2]] + \frac{1}{a}[(a_{1\phi})\pi[\Pi^2] + (\dot{\phi}a_{1X})\dot{\pi}[\Pi^2] + (-2H\dot{\phi}a_{1X})\mathcal{L}^{Gal}_3 + (a_1 - 2Xa_{1X})\Phi[\Pi^2] + (2a_1)\Psi\mathcal{E}^{Gal}_3 + (a_1)\Psi[\Pi^2] + (2a_1)(\nabla_i\Psi)(\nabla_j\pi)[\Pi^{ij}]] \quad (3.10)$$

$$\mathcal{L}^{(2)}_{1,Eff} = a[(7H^2a_1 + 4\dot{H}a_1 + 5H^2Xa_{1X} + 2\dot{H}Xa_{1X} + 7H\dot{X}a_{1X} + \frac{1}{2}\ddot{\phi}^2a_{1X} + \ddot{X}a_{1X} + 2HX\dot{X}a_{1XX} + \dot{X}^2a_{1XX} + \ddot{\phi}a_{1\phi} + 2H\dot{\phi}Xa_{1\phi X} + 3H\dot{\phi}a_{1\phi} + 4X\ddot{\phi}a_{1\phi X} + 2Xa_{1\phi\phi})\pi\nabla^2\pi - (2a_1)\ddot{\pi}\nabla^2\pi + (6H\dot{\phi}a_1 + 6\ddot{\phi}a_1 + 4H\dot{\phi}Xa_{1X} + 4\dot{X}\dot{\phi}a_{1X} + 8Xa_{1\phi})\Phi\nabla^2\pi + (4\dot{\phi}a_1)\dot{\Phi}\nabla^2\pi + (2\dot{\phi}a_1)\dot{\Psi}\nabla^2\pi + (4Xa_1)\Phi\nabla^2\Phi] \quad (3.11)$$

3.1.3 $\sqrt{-g}a_2[\phi, X]L_2$

$$\mathcal{L}^{(NL)}_{2,Eff} = \frac{1}{a^3}[(-\frac{a_{2X}}{2})(\nabla\pi)^2[\Pi]^2] + \frac{1}{a}[(a_{2\phi})\pi[\Pi]^2 + (\dot{\phi}a_{2X})\dot{\pi}[\Pi]^2 + (-6H\dot{\phi}a_{2X} - 2\ddot{\phi}a_{2X})\mathcal{L}^{Gal}_3 + (-2a_2)\Phi\mathcal{E}^{Gal}_3 + (a_2 - 2Xa_{2X})\Phi[\Pi]^2 + (2a_2)(\nabla_i\Phi)(\nabla_j\pi)[\Pi^{ij}] + (2a_2)\Psi\mathcal{E}^{Gal}_3 + (a_2)\Psi[\Pi]^2 + (-2a_2)(\nabla_i\Psi)(\nabla_j\pi)[\Pi^{ij}]] \quad (3.12)$$

$$\mathcal{L}^{(2)}_{2,Eff} = a[(4X\ddot{\phi}a_{2X} + 12H\dot{\phi}Xa_{2X} + 4\ddot{\phi}a_2 + 12H\dot{\phi}a_2)\Phi\nabla^2\pi + (4H^2a_2 + 3\dot{H}a_2 + 15H^2Xa_{2X} + 6\dot{H}Xa_{2X} + 13H\dot{X}a_{2X} + \frac{1}{2}\ddot{\phi}^2a_{2X} + \ddot{X}a_{2X} + 6HX\dot{X}a_{2XX} + \dot{X}^2a_{2XX} - 3H\dot{\phi}a_{2\phi} - 2\ddot{\phi}a_{2\phi} + 6H\dot{\phi}Xa_{2\phi X} + 2X\ddot{\phi}a_{2\phi X})\pi\nabla^2\pi + (2\dot{\phi}a_2)\dot{\Phi}\nabla^2\pi + (6\dot{\phi}a_2)\dot{\Psi}\nabla^2\pi + (-2a_2)\ddot{\pi}\nabla^2\pi] \quad (3.13)$$

3.1.4 $\sqrt{-g}a_3[\phi, X]L_3$

$$\mathcal{L}^{(NL)}_{3,Eff} = \frac{1}{a^3}[(a_3)\mathcal{L}^{Gal}_4 + (a_3)(\nabla_i\pi)(\nabla_j\pi)[\Pi^2]^{ij}] + \frac{1}{a}[(-2\dot{\phi}a_3)(\nabla_i\dot{\pi})(\nabla_j\pi)[\Pi]^{ij} + (-3H\dot{\phi}a_3 - 3\ddot{\phi}a_3 - 2X\ddot{\phi}a_{3X} - 4Xa_{3\phi})\mathcal{L}^{Gal}_3 + (-2Xa_3)\Phi\mathcal{E}^{Gal}_3 + (2Xa_3)(\nabla_i\Phi)(\nabla_j\pi)[\Pi]^{ij}] \quad (3.14)$$

$$\mathcal{L}^{(2)}_{3,Eff} = a[(12H^2Xa_3 + 6\dot{H}Xa_3 + 7H\dot{X}a_3 + 2H\dot{X}Xa_{3X} - 3X\ddot{\phi}^2a_{3X} + \\ - X\ddot{X}a_{3X} - X\dot{X}^2a_{3XX} + 6H\dot{\phi}Xa_{3\phi} + 2X\ddot{\phi}a_{3\phi} - \dot{\phi}X\dot{X}a_{3\phi X})\pi\nabla^2\pi + \\ (2Xa_3)\ddot{\pi}\nabla^2\pi + (-2\dot{\phi}Xa_3)\dot{\Phi}\nabla^2\pi + (6H\dot{\phi}Xa_3 - 6X\ddot{\phi}a_3 - 2\dot{\phi}X\dot{X}a_{3X})\Phi\nabla^2\pi] \tag{3.15}$$

3.1.5 $\sqrt{-g}a_4[\phi, X]L_4$

$$\mathcal{L}^{(NL)}_{4,Eff} = \frac{1}{a^3}[(a_4)(\nabla_i\pi)(\nabla_j\pi)[\Pi^2]^{ij}] + \frac{1}{a}[(4Xa_4)(\nabla_i\Phi)(\nabla_j\pi)[\Pi]^{ij} + (-2\dot{\phi}a_4)(\nabla_i\dot{\pi})(\nabla_j\pi)[\Pi]^{ij}] \tag{3.16}$$

$$\mathcal{L}^{(2)}_{4,Eff} = a[(-4X^2a_4)\Phi\nabla^2\Phi + (-4H\dot{\phi}Xa_4 - 12X\ddot{\phi}a_4 - 4\dot{\phi}X\dot{X}a_{4X} - 8X^2a_{4\phi})\Phi\nabla^2\pi + \\ (\dot{X}Ha_4 - H^2Xa_4 - \dot{H}Xa_4 - 2HX\dot{X}a_{4X} - 3X\ddot{\phi}^2a_{4X} - X\ddot{X}a_{4X} - X\dot{X}^2a_{4XX} + \\ -2H\dot{\phi}Xa_{4\phi} - 3X\ddot{\phi}a_{4\phi} - 4X^2\ddot{\phi}a_{4\phi X} - 2X^2a_{4\phi\phi})\pi\nabla^2\pi + (-4\dot{\phi}Xa_4)\dot{\Phi}\nabla^2\pi + (2Xa_4)\ddot{\pi}\nabla^2\pi] \tag{3.17}$$

3.1.6 $\sqrt{-g}a_5[\phi, X]L_5$

$$\mathcal{L}^{(NL)}_{5,Eff} = \frac{1}{a}[(4X\ddot{\phi}a_5)\mathcal{L}^{Gal}_3] \tag{3.18}$$

$$\mathcal{L}^{(2)}_{5,Eff} = a[(-8X^2\ddot{\phi}a_5)\Phi\nabla^2\pi + (-8HX\dot{X}a_5 - 8X\ddot{\phi}^2a_5 - 4X\ddot{X}a5 + \\ - 3X\dot{X}^2a_{5X} - 8X^2\ddot{\phi}a_{5\phi})\pi\nabla^2\pi] \tag{3.19}$$

3.1.7 $\sqrt{-g}P[\phi, X]$

$$L^{(2)}_{P,eff} = \frac{P_X}{2}\pi\nabla^2\pi \tag{3.20}$$

3.1.8 $\sqrt{-g}\mathcal{L}_m$

$$L^{eff}_m = -\frac{a^3\rho_m}{\widetilde{M}_{Pl}}\Phi\delta \tag{3.21}$$

3.2 DHOST Effective Action

To sum up the previous results in a more compact formula, we can write

$$\mathcal{L}^{(2)}_{DHOST} = a[F\Psi\nabla^2\Psi + G\Psi\nabla^2\Phi + K\Phi\nabla^2\Phi + \eta\pi\nabla^2\pi + \eta_t\ddot{\pi}\nabla^2\pi \\ \xi_1\Phi\nabla^2\pi + \xi_2\Psi\nabla^2\pi + \chi_1\dot{\Phi}\nabla^2\pi + \chi_2\dot{\Psi}\nabla^2\pi] - \frac{a^3\rho_m}{\widetilde{M}_{Pl}}\Phi\delta \tag{3.22}$$

for the quadratic operators, and

$$\begin{aligned}\mathcal{L}^{(NL)}_{DHOST} =& \frac{1}{a^3\Lambda^6}[\nu\mathcal{L}_4^{Gal} + \theta_1(\nabla\pi)^2[\Pi]^2 + \theta_2(\nabla\pi)^2[\Pi^2] + \sigma\nabla_i\pi\nabla_j\pi[\Pi^2]^{ij}] \\ &+ \frac{1}{a\Lambda^3}[\mu\mathcal{L}_3^{Gal} + \mu_t\nabla_i\dot{\pi}\nabla_j\pi\Pi^{ij} + \alpha_1\Phi\mathcal{E}_3^{Gal} + \alpha_2\Psi\mathcal{E}_3^{Gal} + \alpha_*\nabla_i\Psi\nabla_j\pi\Pi^{ij} \\ &+ \beta_*\nabla_i\Phi\nabla_j\pi\Pi^{ij} + p_1\pi[\Pi]^2 + p_{1t}\dot{\pi}[\Pi]^2 + p_2\pi[\Pi^2] + p_{2t}\dot{\pi}[\Pi^2] \\ &+ f_1\Phi[\Pi^2] + f_2\Phi[\Pi]^2 + g_1\Psi[\Pi^2] + g_2\Psi[\Pi]^2]\end{aligned} \tag{3.23}$$

for the nonlinear terms, instead. An additional mass scale, Λ, typical of the effective theory now appeares explicitly.
Compared to the result showed in [1], in which the EFT for a combination of Hordenski and beyond Hordenski terms is listed, a set of new operators appear when dealing with the EFT for a quadratic DHOST theory:

$$\begin{gathered}\{\nabla_i\pi\nabla_j\pi[\Pi^2]^{ij}, \nabla_i\Phi\nabla_j\pi[\Pi]^{ij}, (\nabla\pi)^2[\Pi]^2, (\nabla\pi)^2[\Pi^2], \nabla_i\pi\nabla_j\pi[\Pi]^{ij}, \\ \pi[\Pi]^2, \dot{\pi}[\Pi]^2, \pi[\Pi^2], \dot{\pi}[\Pi^2], \Phi[\Pi^2], \Phi[\Pi]^2, \Psi[\Pi^2], \Psi[\Pi]^2, \Phi\nabla^2\Phi, \dot{\Phi}\nabla^2\pi, \ddot{\pi}\nabla^2\pi\}\end{gathered} \tag{3.24}$$

The goal of this thesis work is now clearer: first we need to retrieve from this effective action the results presented in [1]. To do so we will impose the relations between the coefficients that appear in (3.22) and (3.23) that select the combination of quartic Horndenski lagrangians. Then we will derive the equations of motion of the perturbations and manipulate them in spherical cohordinates and under the spherical symmetry assumption: with this procedure we will solve them and then we will study the behavior of the solution to check the integrity of the Vainshtein mechanism.
Second step, consequently, will then be to redo the same analysis in a more general case, the study of the effective theory to understand what is the effect of these new nonlinear operators arising in the generalized case of considering a wider class of DHOST theories. What we expect is that the effect of these operators will reproduce similar effects as seen those already seen in the case of the beyond Horndeski terms, effectively breaking the Vainshtein screening mechanism inside the region in which it would be expected to work.

3.3 ET for Class I DHOST theories

We will choose to restrict to the case of a DHOST theory with the additional constraint $a_1 = -a_2$: in this way, the contributions of some of the new operators are canceled or reabsorbed in the coefficient of other operators. This is the condition that defines the *class I* DHOST theories, without specifying the actual degenerate subclass.

We can see that, for example, as a result of the addition of this constraint, $p_1 = -p_2$ and so the operators proportional to these coefficients sum up to a term that is equivalent, once integrated by parts, to the $\mathcal{L}_4^{Gal}$ operator. This is still a more general case than the one studied in [1] and showed in the previous section, but we exploit the simplification that this relation between DHOST coefficients brings in. Eventually, the result, where the $\{A_i, B_i, C_i\}$ coefficients have been introduced for simplicity, is:

$$\begin{aligned}\mathcal{L}^{eff}_{a_1=-a_2} &= a(A_1\Phi\nabla^2\pi + A_2\Psi\nabla^2\pi + A_3\pi\nabla^2\pi + A_4\Psi\nabla^2\Phi + A_5\Psi\nabla^2\Psi \\ &+ A_6\dot{\Psi}\nabla^2\pi + A_7\Phi\nabla^2\Phi + A_8\dot{\Phi}\nabla^2\pi + A_9\ddot{\pi}\nabla^2\pi) \\ &+ \frac{1}{a\Lambda^3}(B_1\mathcal{L}_3^{Gal} + B_2\Phi\mathcal{E}_3^{Gal} + B_3\Psi\mathcal{E}_3^{Gal} \\ &+ B_4\nabla_i\Psi\nabla_j\pi\Pi^{ij} + B_5\nabla_i\Phi\nabla_j\pi\Pi^{ij} + B_6\nabla_i\dot{\pi}\nabla_j\pi\Pi^{ij}) \\ &+ \frac{1}{a^3\Lambda^6}(C_1\mathcal{L}_4^{Gal} + C_2\nabla_i\pi\nabla_j\pi[\Pi^2]^{ij}) - \frac{a^3\Phi}{\widetilde{M}_{Pl}}\delta\rho_m\end{aligned} \tag{3.25}$$

where the conversion table from the $\{A_i, B_i, C_i\}$ dimensionful coefficients to the original $\{f, a_i, P\}$ DHOST coefficients is given in the appendix A.

3.4 Equations of motion: Class I

From the effective lagrangian built in the previous section we would like now to derive the equations of motion of the perturbations, by variating $\mathcal{L}^{eff}_{a_1=-a_2}$ with respect to $\{\Phi, \Psi, \pi\}$. By doing so, we obtain three equations that can be re-expressed in spherical coordinates and assuming spherically symmetry, can be integrated once to obtain

δ_Φ :

$$A_1x + 2A_7y + A_4z - \dot{A}_8x - 3A_8Hx - A_8\dot{x} - 2\Lambda^3x(-2B_2x + B_5(x + rx')) = A(r,t) \tag{3.26}$$

δ_Ψ :

$$A_2x + A_4y + 2A_5z - \dot{A}_6x - 3A_6Hx - A_6\dot{x} - 2\Lambda^3x(-2B_3x + B_4(x + rx')) = 0 \tag{3.27}$$

δ_π :

$$\begin{aligned}&2A_3x + A_1y + 2A_8Hy + A_2z + 2A_6Hz + 6\dot{A}_9Hx + \ddot{A}_9x + 2\dot{A}_9\dot{x} + A_8\dot{y} + A_6\dot{z} \\ &+ A_9(13H^2x + 5\dot{H}x + 10H\dot{x} + 2\ddot{x}) + 2\Lambda^3(2B_1x^2 + B_6(r\dot{x}x' + Hx(9x + 5rx')) \\ &+ x(5\dot{x} + 2r\dot{x}')) + x(4\Lambda^3C_1x^2 + 4B_2y + 3B_5y + 4B_3z + 3B_4z + \dot{B}_6x + \dot{B}_6rx' \\ &+ B_5ry' + B_4rz' + 4\Lambda^3C_2(3x^2 + r^2x'^2 + rx(6' + rx'')))) = 0\end{aligned} \tag{3.28}$$

Here we introduced the new unknown functions x,y,z related to the original perturbations through the definitions

$$x(t,r) \equiv (\frac{\pi'}{\Lambda^3 a^2 r}) \quad y(t,r) \equiv (\frac{\Phi'}{\Lambda^3 a^2 r}) \quad z(t,r) \equiv (\frac{\Psi'}{\Lambda^3 a^2 r}) \quad A(t,r) \equiv \frac{M(r,t)}{\widetilde{M}_{Pl}\Lambda^3 8\pi r^3} \tag{3.29}$$

and moreover,

$$M(t,r) \equiv \int_0^r d\bar{r} 4\pi \bar{r}^2 \rho_m(t)\delta(t,\bar{r}) \tag{3.30}$$

is the source mass, contained in a sphere of comoving radius r and with the prime notation we here refer in short to the derivatives with respect to the comoving radial coordinate, $' \equiv \frac{d}{dr}$. The dot notation instead is a shortcut for usual cosmological time derivatives, $\dot{} \equiv \frac{d}{dt}$.

3.5 ET for quartic Horndeski

We can also easily retrieve the effective theory for the theory described by the combination of the two quartic Horndeski and beyond Horndeski lagrangians by imposing the conditions (2.71) to the results of the perturbation procedure, (3.8) to (3.19). The effective lagrangian thus obtained is equivalent to the one showed in [1], without the contributions offered by the quintic Horndeski terms

$$\mathcal{L}_4^{eff} = \mathcal{L}_H^{(2)} + \mathcal{L}_{bH}^{(2)} + \mathcal{L}_H^{NL} + \mathcal{L}_{bH}^{NL} \tag{3.31}$$

where the quadratic lagrangians are described as

$$\mathcal{L}_H^{(2)} = -a\mathcal{F}\Psi\nabla^2\Psi + 2a\mathcal{G}\Psi\nabla^2\Phi + \frac{a\eta}{2}\pi\nabla^2\pi - 2a\xi_1\Phi\nabla^2\pi + 4a\xi_2\Psi\nabla^2\pi - a^3\rho_m\Phi\delta \tag{3.32}$$

$$\mathcal{L}_{bH}^{(2)} = \frac{4a\xi_t}{\Lambda^3}\dot{\Psi}\nabla^2\pi \tag{3.33}$$

and the nonlinear lagrangians are

$$\mathcal{L}_H^{NL} = \frac{\mu}{a\Lambda^3}\mathcal{L}_3^{Gal} + \frac{\nu}{a^3\Lambda^6}\mathcal{L}_4^{Gal} + \frac{\alpha_1}{a\Lambda^3}\Phi\mathcal{E}_3^{Gal} + \frac{\alpha_2}{a\Lambda^3}\Psi\mathcal{E}_3^{Gal} \tag{3.34}$$

$$\mathcal{L}_{bH}^{NL} = -\frac{4\alpha_*}{a\Lambda^3}\nabla_i\Psi\nabla_j\Pi^{ij} \tag{3.35}$$

the $\{\mathcal{F}, \mathcal{G}, \eta, \xi_1, \xi_2, \xi_t, \mu, \nu, \alpha_1, \alpha_2, \alpha_*\}$ dimensionless coefficients are functions of the Horndeski coefficients $\{G_4, G_{4X}, F_4, F_{4X}\}$, and their definition can be found in [1], [8] without the contributions of $\{G_5, G_{5X}, F_5, F_{5X}\}$.

3.6 Equations of motion:$\mathcal{L}_4^H + \mathcal{L}_4^{bH}$

The procedure here is the same as the one already shown in the class I case, but as we will see the result is way simpler to handle in order to be able to derive a physical interpretation.
Indeed, the generalization to a wider class of DHOST brings into play many additional terms, among which we encounter some featuring up to second order time and radial derivatives. Of course, these terms complicate the solution of the eom, but in the next chapter we will see that in a particular regime of interest part of these terms are neglegible.
After the derivation of the eom and their integration under the spherical symmetry assumption, we obtain the following equations:

$$\begin{aligned} &\delta_\Phi : \\ &\mathcal{G}z - \xi_1 x - \alpha_1 x^2 = A \end{aligned} \tag{3.36}$$

$$\begin{aligned} &\delta_\Psi : \\ &2\xi_2 x + \mathcal{G}y - \mathcal{F}z + \alpha_2 x^2 + 2\alpha_* x(rx' + x) - \frac{2}{a^3}\partial_t(a^3 \xi_t x) = 0 \end{aligned} \tag{3.37}$$

$$\begin{aligned} &\delta_\pi : \\ &\eta x - 2\xi_1 y + 4\xi_2 z + 2\mu x^2 + 2\nu x^3 - 4\alpha_1 xy + 4\alpha_2 xz - 4\alpha_*(rxz' + 3xz) \\ &+ \frac{4\xi_t}{a^2}\partial_t(a^2 z) = 0 \end{aligned} \tag{3.38}$$

We will deal with the solution of these equations and of the equations in the previous section in the next chapter, when we will eventually complete the analysis of the Vainshtein mechanism in both these classes of ST theories.

Chapter 4

Screening

As anticipated in the introduction, modified gravity models require all some sort of censorship around those scales for which the last century of observational physics has provided us with precision tests. These tests General Relativity has endured up to now, and this fact contributes to the reasonable convintion with which we do not expect effects of physics beyond GR to arise around scales such as the solar system size. What is needed in modified gravity, thus in particular in the ST theories we study in this work, is a screening mechanism that hides on small scales the propagation of the additional degrees of freedom introduced to extend GR, while allowing for significant deviantions from Einstein's phenomenology on cosmological scales.
In this chapter we will introduce a classification of the main type of screening mechanisms; then we will deal with the Vainshtein mechanism and a simple example in which this artefact works. In the next chapter, finally, we will build the effective theory for small scales, with which we will study the Vainsthein screening in $L_4^H + L_4^{bH}$, via the more general qDHOST lagrangian.

4.1 Zoology of screening mechanisms

The classification is here presented as was introduced in [6]: it is a phenomenological classification based on the criterion around which the mechanism themselves work. In general, the operating principle is that a certain operator in the lagrangian describing the MG theory becomes more relevant on certain scales than the others and by doing so, kill the propagation of the expected additional d.o.f.
There are three cases to distinguish:

1. *ϕ-based screening*: the screening mechanisms in this class all operate on the principle that the fifth force is screened in regions of high density, ρ, or equivalently, high Newtonian potential, Φ. These mechanisms all arise from an effective potential, built with contributions

from the self-interaction potential of the scalar field, $V(\phi)$, and the non minimal coupling between the scalar and the matter sector, $A(\phi)$, from which emerges a dependance on the local density. In the case of the chameleon screening, this is the key to the mechanism: in context of high density regions, the scalar acquires a consistent mass, hindering the propagation. In low density media, instead, the effective potential is shallower and thus the effective mass lighter, allowing for the fifth force to propagate.
In the symmetron or dilaton mechanisms, instead, the ρ-dependance of the effective potential determines a symmetry breaking behavior: in situations of low density, the local field acquires a null vacuum expectation value; in high density environments, instead, a non trivial VEV is restored. By choosing smartly the non minimal coupling, one can demonstrate that the coupling between the scalar fluctuations and the matter sector is proportional to the VEV: low values of density are thus associated to an effective decoupling of fluctuations and matter, coupling that is restored instead in high density regions.

2. *$\partial\phi$-based screening* This is the first class of derivative based screening. This type of mechanisms, also called *kinetic screening* are based on the first order derivatives of the relevant fields. That means they operate when $\partial\phi \gtrsim \Lambda^2$; or, equivalently, when the value of the gradient of the Newton potential exceeds a threshold value, $|\nabla\Phi| \gtrsim \Lambda^2$. Thus the fifth force is shut down when the *gravitational acceleration* overcomes the critical value $\vec{a} = -\vec{\nabla}\Phi$.
Examples of theories featuring the kinetic screening are the *k-mouflage* or, more in general, $P(X)$ models.

3. *$\partial^2\phi$-based screening* The Vainshtein screening mechanism belongs to this class. The principle is that on small scales, terms built with second-order derivatives of the fields become relevant while higher-order terms remain neglegible. The mechanism activates when $\partial^2\phi \gtrsim \Lambda^3$, or equivalently when the curvature terms or local density exceeds the threshold as described by $R \sim |\nabla^2\Phi| \gtrsim \Lambda^3$. The well known example featuring the Vainshtein screening is the galileon theory, as will be better explained in the next section.

4.2 Vainshtein screening: cubic galileon theory

Galileon theories are the most general theory of a scalar field, $\pi(x)$, in flat space-time (thus the metric is non-dynamical), that features only terms not higher than second order in the derivatives of $\pi(x)$. In fact, also terms involving non-derived powers of the scalar field, or terms containing only once derived $\pi(x)$ must be rejected in building galileon theories. The covariantization, in the sense of the extension to curved-space galileons, is represented by the Horndeski ST theory.

In order to show how the Vainshtein mechanism works, we will present here a famous example of the cubic galileon theory:

$$\mathcal{L} = -3(\partial\phi)^2 - \frac{1}{\Lambda^3}\Box\phi(\partial\phi)^2 + \frac{g}{M_{Pl}}\phi T^\mu_\mu \tag{4.1}$$

and the screening acting on it.
The equation of motion for ϕ is

$$6\Box\phi + \frac{2}{\Lambda^3}[(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2] + \frac{g}{M_{Pl}}T^\mu_\mu = 0 \tag{4.2}$$

The Vainshtein screening relies on the non-linear term, $\frac{1}{\Lambda^3}\Box\phi(\partial\phi)^2$, becoming much more relevant than the kinetic term around massive sources, such that $\Box\phi \gg \Lambda^3$.
To actually see this effect, let's take a static point-like source such that $T^\mu_\mu = -M\delta^{(3)}(\vec{x})$. We then further assume that the scalar field profile is spherically symmetric and static as well, $\phi = \phi(r)$, for simplicity. Finally, we integrate once the equation (4.2) under these assumptions to obtain

$$6\phi' + \frac{4}{\Lambda^3}\frac{\phi'^2}{r} = \frac{gM}{4\pi r^2 M_{Pl}} \tag{4.3}$$

This is a simple second order equation, and to select the physically relevant solution between the two we will just impose the reasonable condition for the scalar field's radial derivative $\phi' \to 0$ at spatial infinity. This way we select the solution

$$\phi'(r) = \frac{3\Lambda^3 r}{4}\Big(-1 + \sqrt{1 + \frac{1}{9\pi}(\frac{r_V}{r})^3}\Big) \tag{4.4}$$

and we define the peculiar *Vainshtein radius* as $r_V \equiv \frac{1}{\Lambda}(\frac{gM}{M_{Pl}})^{\frac{1}{3}}$. This is a specific distance scale introduced for the screening mechanism, and has the role of a threshold that distinguishes between two regimes:

- $r \gg r_V$, far outside the Vainshtein typical radius, the solution acquires the form
$$\phi' \simeq \frac{g}{3}\frac{M}{8\pi M_{Pl} r^2}$$
Hence, the galileon force to gravitational force ratio is approximately
$$\frac{F_\phi}{F_{grav}}\Big|_{r\gg r_V} \simeq \frac{g^2}{3}$$

- $r \ll r_V$, inside the Vainshtein radius, instead, we can see that the galileon force is highly suppressed. The profile being,

$$\phi' \simeq \frac{\Lambda^3 r_V}{2}\sqrt{\frac{r_V}{r}}$$

and the ration between the forces resulting in

$$\frac{F_\phi}{F_{grav}}\Big|_{r\ll r_V} \simeq (\frac{r_V}{r})^{\frac{3}{2}} \ll 1$$

The effective theory built from scalar tensor theories much more general than this galileon example (aka Horndeski [9]) still feature nonlinear operators that behave similarly as seen in this section, with the introduction of a threshold distance scale that distinguishes between screened and unscreened regimes.
However, is has been shown that even the simplest generalization of Horndeski theory seems to introduce higher order nonlinear operators that interfere with the Vainshtein mechanism and thus impede the screening. In the following chapter, we will deal with what has been already shown in this direction by [1] and we will set the foundations of the argument that goeas in the direction of generalizing this proof to a wider class of ST theories.

4.3 Breaches in the screening mechanism

In this section we will finally focus on cases in which the Vainshtein screening is broken: the Horndenski plus beyond Horndeski theory and a more generalized DHOST case.
The effect of nonlinear lagrangian operators on small scales becomes relevant, and implies the appearance of second order derivatives in r and time derivatives: the solutions of the eom for the metric potentials in general is expected to hide some unpleasant surprises, in the form of $\Phi \neq \Psi$ even for some small scale regime, thus signaling the presence of beyond GR effects where we wouldn't like them to arise.
Here follows the detailed solution of the two models presented in the previous chapter.

4.3.1 Vainshtein breaking in $\mathcal{L}_4^H + \mathcal{L}_4^{bH}$

In dorder to solve equations (3.36)-(3.38) we will first select the nonlinear regime corresponding to $A \gg 1$. This regime is equivalent to selecting the proximity limit to the source, in which we expect the nonlinearities in our theory to gain more significance and breaking effects to show up.
The procedure here follows closely the one adopted in [1]: we make use of equations (3.36) and (3.37) to eliminate y and z from (3.38).

In this simple case, the time and radial derivatives of x disappear in a seemingly miracolous fashion. We are left with an ordinary third grade equation in x, where the only complications hide in the explicit form of its coefficients

$$\begin{aligned}&\Xi x^3 - \kappa_2 x^2 - 2[\kappa_1 + (\mathcal{F}\alpha_1 - \mathcal{G}\alpha_2 + 3\mathcal{G}\alpha_*)A + \mathcal{G}\alpha_* rA']x \\ &+ [\frac{2\mathcal{G}^2\xi_t}{\mathcal{M}a^2}\partial_t(\frac{a^2}{\mathcal{G}}A) - (\mathcal{F}\xi_1 - 2\mathcal{G}\xi_2)] = 0\end{aligned} \tag{4.5}$$

The explicit form of the coefficients being

$$\kappa_1 \equiv \frac{-\mathcal{G}^2\eta}{2} + \frac{\mathcal{F}\xi_1^2}{2} - 2\mathcal{G}\xi_1\xi_2 - \frac{G\xi_t^2}{a\mathcal{M}}\partial_t(\frac{a\xi_1}{\mathcal{G}\xi_t}) \tag{4.6}$$

$$\kappa_2 \equiv 3\mathcal{F}\alpha_1\xi_1 - \mathcal{G}(\mathcal{G}\mu + 3\alpha_2\xi_1 + 6\alpha_1\xi_2 - 4\alpha_*\xi_1) - \frac{2a^4\mathcal{G}\xi_t^3}{\mathcal{M}}\partial_t(\frac{\alpha_1}{a^4\mathcal{G}\xi_t^2}) \tag{4.7}$$

$$\Xi \equiv \mathcal{G}(4\alpha_1\alpha_2 - 2\alpha_1\alpha_* + \mathcal{G}\nu) - 2\mathcal{F}\alpha_1^2 \tag{4.8}$$

Furthermore, we look for solution that are $x \gg 1$: for such, we may expect (in the absence of the beyond Honrdenski terms) the existence of a Vainshtein radius inside of which ordinary GR is reproduced.
Considering $A \sim \epsilon$ and $x \sim \sqrt{\epsilon}$ we can conduct an expansion in the ϵ parameter and keep only leading order terms in it. In doing so, we will assume that $rA' \sim O(A)$. The result of this limit is the approximate solution

$$x^2 \simeq \frac{2[(\mathcal{F}\alpha_1 - \mathcal{G}\alpha_2 + 3\mathcal{G}\alpha_*)A + \mathcal{G}\alpha_* rA']}{\Xi} + O(A^{1/2}) \tag{4.9}$$

Now, by taking the same limit in equations (3.36) and (3.37), thus neglecting the terms that are of lower order than $O(A)$, such as linear terms in x we obtain the following approximate equations

$$z \simeq \frac{\alpha_1}{\mathcal{G}}x^2 + A \tag{4.10}$$

$$y \simeq \frac{1}{\mathcal{G}}[\mathcal{F}z - \alpha_2 x^2 - \alpha_* r\frac{d}{dr}(x^2) - 2\alpha_* x^2] \tag{4.11}$$

All that is left to do now is to substitute (4.9) in these two equations to obtain

$$y = 8\pi G_N A - \frac{2\alpha_*^2}{\Xi}\frac{(r^3A)''}{r} \tag{4.12}$$

$$z = 8\pi G_N A + \frac{2\alpha_1\alpha_*}{\Xi}\frac{(r^3A)'}{r^2} \tag{4.13}$$

where G_N stands for the effective Newton constant, that actually has a time dependence inherited from the Horndeski coefficients G_4, F_4 and their derivatives with respect to X, of which G_N is a function defined as:

$$8\pi G_N(t) \equiv [2G_4 - 8X(G_{4X} + XG_{4XX}) - 4X^2(5F_4 + 2XF_{4X})]^{-1} \tag{4.14}$$

This relation can be derived from the background equations of motion, which are reproduced in the appendix A.

The two solutions for the metric potentials share a common term, hence it is clear that if there is a limit in which the second terms are neglegible, then in that limit we retrieve simple GR. Yet, it still has to be proven that the time dependancy of $G_N(t)$ is such that it varies sufficiently slow to effectively reproduce a universal constant.
We choose to take a density profile such that the mass source is confined in a limited region, to which we will refer from now on as *ovedensity region*, determined by a specific radius, $r_s(t)$. The density fluctuations are considered to vanish for all the points that $r > r_s(t)$.
We can then define a Vainshtein radius as

$$r_V(t) \equiv [\frac{M(t, r_s)}{8\pi \tilde{M}_{Pl} \Lambda^3}]^{1/3} \tag{4.15}$$

We implicitly assume here that the overdensity region is enclosed in the Vainshtein covered region: $r_s < r_V$. Consequently we see that outside the confinement region we have

$$A = (\frac{r_V}{r})^3 \tag{4.16}$$

Instead the first and second derivatives of the quantity $A(t, r)$ vanish in this limit, since it is true that

$$(r^3 A)' \propto M(t, r)' \propto r^2 \delta = 0$$

For this reason the second terms that distinguish the potentials one from another vanish for $r > r_s$, so that

$$\frac{\Phi}{\tilde{M}_{Pl}} = \frac{\Psi}{\tilde{M}_{Pl}} = -\frac{G_N(t) M(t, r_s) a^2}{r} \tag{4.17}$$

This is the ordinary Newtonian potential, as soon as we assume that $G_N(t) \sim G_N$, for slow varying effective constants.

Nonetheless, it is clear that the Vainshtein mechanism ceases to work inside the overdensity region, $r < r_s$, where the terms proportional to A' and A'' do not vanish anymore, in general. In the overdensity region, finally, we have

$$\Phi \neq \Psi$$

thus signaling the presence of modified gravity effects to which observational tests could be sensible, in principle.

We observe that in the limit $\alpha_* \to 0$, that is in the limit in which beyond Horndeski terms are turned off, we reactivate the screening mechanism, as the two terms responsible for the breaking vanish. This prooves that the actual term responsible for the rupture of the Vainshtein is the nonlinear effective operator (3.35).

4.3.2 Vainshtein breaking in Class I

The manipulations required in this case in order to solve the eom are slightly more complex. As was already pinpointed before, the presence of second order time and radial derivatives, not to mention mixed derivatives, of the unknown function x,y and z complicate the solution of the system of equations (3.26) to (3.28). It is impossible to recover the same simplification of the time and radial derivatives by simple substitution of y and z exploting the first two equations and substituing in the third one.
Nevertheless, there are some manipulations that we can conduct in order to ease the task of solving the eom.

First of all, we are still interested in the same vicinity limit to the source to which we restricted our search for a solution previously. This translates again in the limits $A \gg 1$ and $x \gg 1$.
In this limit we know already that the nonlinear terms are expected to gain in importance over other neglegible terms. In fact, we can look again for solutions that have a defined gerarchy in terms of the function $A(t, r)$: again we could search for $x \sim O(A^{1/2})$ and $y, z \sim O(A)$, and throw away all the terms in the eom that are of lower order in A.
In order to implent this argument, we can operate the following rescalings

$$x \to x\epsilon \quad y \to y\epsilon^2 \quad z \to z\epsilon^2 \quad A \to A\epsilon^2 \tag{4.18}$$

and then take only the leading order of the expansion in terms of ϵ of (3.26), (3.27) and (3.28), for big values of ϵ. The result of this step is

$$A_4 y + 2A_5 z + 2\Lambda^3(2B_3 - B_4)x^2 - \Lambda^3 r B_4 (x^2)' - A = 0 \tag{4.19}$$

$$2A_7 y + A_4 z + 2\Lambda^3(2B_2 - B_5)x^2 - \Lambda^3 r B_5 (x^2)' = 0 \tag{4.20}$$

$$\begin{aligned} &8\Lambda^3(C_1 + 3C_2)x^3 + 2(4B_2 + 3B_5)xy + 2(4B_3 + 3B_4)xz + 16\Lambda^3 r C_2 (x^3)' \\ &+ 8\Lambda^3 r^2 C_2 x (x')^2 + 8\Lambda^3 r^2 C_2 x^2 x'' + 2rB_5 x y' + 2rB_4 x z' = 0 \end{aligned} \tag{4.21}$$

In the first two equations the LO is represented by $O(\epsilon^2)$ terms, while in the third one the terms $O(\epsilon^3)$ terms are LO.
Before proceeding with the (y, z) substitution in (4.21), we shall first notice that there is a change of variables that could lead us to a linear system of equations which is still equivalent to this one

$$\begin{cases} x(t,r) \equiv \sqrt{\frac{X(t,r)}{r^3}} \\ y(t,r) \equiv \frac{Y(t,r)}{r^3} \\ z(t,r) \equiv \frac{Z(t,r)}{r^3} \\ A(t,r) \equiv \frac{M(t,r)}{r^3} \end{cases} \tag{4.22}$$

Thanks to the substitutions (4.22) the system of equations is now linear in the new unknown functions $\{X, Y, Z\}$, and features only radial derivatives at most of the second order

$$\frac{2A_7}{r^3}Y + \frac{A_4}{r^3}Z + \frac{4\Lambda^3 B_2 + B_5}{r^3}X - \frac{\Lambda^3 B_5}{r^2}X' = \frac{M}{r^3} \tag{4.23}$$

$$\frac{A_4}{r^3}Y + \frac{2A_5}{r^3}Z + \frac{4\Lambda^3 B_3 + B_4}{r^3}X - \frac{\Lambda^3 B_4}{r^2}X' = 0 \tag{4.24}$$

$$\frac{4B_2}{r^2}Y + \frac{4B_3}{r^2}Z + \frac{4\Lambda^3 C_1}{r^2}X + \frac{B_5}{r}Y' + \frac{B_4}{r}Z' + 2\Lambda^3 C_2 X'' = 0 \tag{4.25}$$

From this point we can now obtain a master equation in X, which will result to be a non-homogeneous second-order Euler-Cauchy equation of the form

$$T(t)X''(t,r) + R(t)\frac{X(t,r)}{r^2} + S_1(t)\frac{M(t,r)}{\Lambda^3 r^3} + S_2(t)\frac{M'(t,r)}{\Lambda^3 r^2} = 0 \tag{4.26}$$

The explicit expression of the coefficients in this equation can be found in the appendices.
The solution to the homogeneous equation results in the combination

$$K_1(t)r^{\frac{1}{2}(1-\zeta(t))} + K_2(t)r^{\frac{1}{2}(1+\zeta(t))} \tag{4.27}$$

Where we defined, for simplicity

$$\zeta(t) \equiv \sqrt{1 - \frac{4R(t)}{T(t)}} \tag{4.28}$$

The particular solution, instead, can be written as

$$\begin{aligned} &P_+(t)\int_1^r \frac{\bar{r}^{\frac{1}{2}[1+\zeta(t)]}}{T(t)\zeta(t)}(S_1(t)\frac{M(\bar{r},t)}{\Lambda^3\bar{r}^2} - S_2(t)\frac{M'(\bar{r},t)}{\Lambda^3\bar{r}})d\bar{r} \\ &- P_-(t)\int_1^r \frac{\bar{r}^{\frac{1}{2}[1-\zeta(t)]}}{T(t)\zeta(t)}(S_1(t)\frac{M(\bar{r},t)}{\Lambda^3\bar{r}^2} - S_2(t)\frac{M'(\bar{r},t)}{\Lambda^3\bar{r}})d\bar{r} \end{aligned} \tag{4.29}$$

The solutions obtained for the master equation, thus, are hard to interpret and give a physical meaning. This is obviously due to the complexity of the maste equation itself. In fact, what can be already observed from the is that the generalization to class I theories brings along a promotion of the master equation to a differential equation, compared to the simple third grade equation obtained in [1] and retrieved a few section before. What we expect is of course to retrieve the Kobayashi solution in the limit of $T(t) \to 0$, and with the proper application of the degeneracy conditions equivalent to the choice of quartic Horndeski + beyond.

Chapter 5

Conclusions

In this project we managed to construct the effective theory for the general quadratic degenerate higher-order scalar-tensor theories on small scales around quasi-static non-relativistic massive sources. The purpose of this ET, being the verification of the Vainshtein mechanism in the proximity of spherically symmetric overdensities.
From this more general starting point, we completed the goal of retrieving the results obtained by *Kobayashi et alii* in [1], which has been done by restricting our ET theory to that one employed in this last mentioned paper. This is a further proof of the actual failure of the Vainshtein mechanism in the Horndeski theory extended with the "beyond" terms introduced in [2].
Finally, we tackled the question on the Vainshtein validity in the generalized case of class I qDHOST theories: we managed to obtain a master equation for the perturbations of the scalar field, which already presents some non trivial effects that do not appear in the simpler case considered in [1]. A deeper analysis of the solution of this equation is postponed to an eventual future pubblication.

Appendices

Appendix A

Equations of motion: background

$$
\begin{aligned}
\delta_{a(t)} :& \\
&9fH^2 - 18a_2H^2X + 6F\dot{H} - 12a_2X\dot{H} - 12a_2H\dot{X} - 12a_{2X}HX\dot{X} + 6f_XH\dot{X} \\
&- \frac{3}{2}a_4\dot{X}^2 + 3a_5X\dot{X}^2 - 3a_{2X}\dot{X}^2 + 3f_{XX}\dot{X}^2 + \frac{3}{2}a_3\dot{\phi}\dot{X} - 3a_2\ddot{X} + 3f_X\ddot{X} = -p
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
\delta_{N(t)} :& \\
&3fH^2 - 12a_{2X}H^2X^2 - 12f_XH^2X + 6a_2\dot{H}X - 6f_X\dot{H}X - 3a_2H\dot{X} - 6a_4HX\dot{X} \\
&+ 12a_5HX^2\dot{X} + 3f_XH\dot{X} + \frac{1}{2}a_4\dot{X}^2 + a_5X\dot{X}^2 - a_{4X}X\dot{X}^2 \\
&+ 2a_{5X}X^2\dot{X}^2 + 3a_3HX\dot{\phi} - 2a_4X^2\ddot{X} + 4a_5X^3\ddot{X} = -\rho
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
\delta_{\phi(t)} :& \\
&9a_3H^2X + 3a_3\dot{H}X + 3a_3H\dot{X} + 3a_{3X}HX\dot{X} + 9a_2H^3\dot{\phi} - 18a_{2X}H^3X\dot{\phi} \\
&- 18f_XH^3\dot{\phi} + 15a_2H\dot{H}\dot{\phi} - 12a_{2X}H\dot{H}X\dot{\phi} - 21f_XH\dot{H}\dot{\phi} + 3a_2H^2\ddot{\phi} \\
&- 9a_4H^2\dot{X}\dot{\phi} + 18a_5H^2X\dot{X}\dot{\phi} - 6a_{2X}H^2\dot{X}\dot{\phi} - 6a_{2XX}H^2X\dot{X}\dot{\phi} \\
&- 6f_XH^2\ddot{\phi} - 6f_{XX}H^2\dot{X}\dot{\phi} + 3a_2\dot{H}\ddot{\phi} - 3a_4\dot{H}\dot{X}\dot{\phi} + 6a_5\dot{H}X\dot{X}\dot{\phi} \\
&+ 3a_{2X}\dot{H}\dot{X}\dot{\phi} - 3f_X\dot{H}\ddot{\phi} - 3f_{XX}\dot{H}\dot{X}\dot{\phi} - 3a_4H\dot{X}\ddot{\phi} + 12a_5H\dot{X}^2\dot{\phi} - \frac{9}{2}a_{4X}\dot{X}^2\dot{\phi} \\
&+ 9a_{5X}HX\dot{X}\dot{\phi} + a_5\dot{X}^2\ddot{\phi} - \frac{1}{2}a_{4X}\dot{X}^2\ddot{\phi} - \frac{1}{2}a_{4XX}\dot{X}^3\dot{\phi} + 5a_{5X}X\dot{X}^2\ddot{\phi} + 2a_{5XX}X^2\dot{X}^2\ddot{\phi} \\
&+ 3a_2\ddot{H}\dot{\phi} - 3f_X\ddot{H}\dot{\phi} - 6a_4H\ddot{X}\dot{\phi} + 12a_5HX\ddot{X}\dot{\phi} - a_4\ddot{X}\ddot{\phi} + 5a_5\ddot{X}\dot{X}\dot{\phi} \\
&- 2a_{4X}\ddot{X}\dot{X}\dot{\phi} + 4a_{5X}X\ddot{X}\dot{X}\dot{\phi} - a_4\dddot{X}\dot{\phi} + 2a_5X\dddot{X}\dot{\phi} = 0
\end{aligned} \tag{A.3}
$$

Appendix B

$\{A_i,B_i,C_i\}$ coefficients

$$\begin{aligned}
A_1 &= -4\dot\phi(HX+\dot X)a_{2X} + 4(3H\dot\phi+\ddot\phi)(Xa_{2X}+a_2) - 8Xa_{2\phi} - 6(H\dot\phi+\ddot\phi)a_2 \\
&\quad + 2X\left[3(H\dot\phi-\ddot\phi)a_3 - \dot X\dot\phi a_{3X}\right] - 4X\left[\dot X\dot\phi a_{4X} + 2Xa_{4\phi}(\phi,X) + a_4(H\dot\phi+3\ddot\phi)\right] \\
&\quad - 8X^2\ddot\phi a_5 + 2(H\dot\phi+\ddot\phi)f_X + 4X\ddot\phi f_{XX} + 4Xf_{\phi X} - 2f_\phi\,, && \text{(B.1)} \\
A_2 &= -4(H\dot\phi+\ddot\phi)f_X - 8X\ddot\phi f_{XX} + 4f_\phi - 8Xf_{\phi X}\,, && \text{(B.2)} \\
A_3 &= 2(2X\dot H + 3H\dot X - 5H^2X)a_{2X} + 4HX\dot Xa_{2XX} - 3(2H\dot\phi+\ddot\phi)a_{2\phi} \\
&\quad - 2(\ddot\phi - 2H\dot\phi)Xa_{2\phi X} - 2Xa_{2\phi\phi} - (\dot H + 3H^2)a_2 - (2H\dot X + \ddot X + 3\ddot\phi^2)Xa_{4X} \\
&\quad - (2H\dot\phi + 3\ddot\phi)Xa_{4\phi} - X\dot X^2a_{4XX} - 4X^2\ddot\phi a_{4\phi X} - 2X^2a_{4\phi\phi} - (X\dot H - H\dot X + H^2X)a_4 \\
&\quad - 3X\dot X^2a_{5X} - 8X^2\ddot\phi a_{5\phi} - 4(2H\dot X + \ddot X + 2\ddot\phi^2)Xa_5 + 3(\dot H + 2H^2)f_X + \frac{P_X}{2}\,, \\
& && \text{(B.3)} \\
A_4 &= 4(f-2Xf_X) = 2M^2(1+\alpha_H)\,, && \text{(B.4)} \\
A_5 &= -2f = -M^2(1+\alpha_T)\,, && \text{(B.5)} \\
A_6 &= 4(a_2 - f_X) = \frac{M^2\dot\phi}{X}\alpha_H\,, && \text{(B.6)} \\
A_7 &= -4X(a_2 + a_4X - f_X) = -\frac{M^2}{2}\beta_3\,, && \text{(B.7)} \\
A_8 &= -2\dot\phi\left[a_2 + (a_3+2a_4)X - f_X\right] = -\frac{M^2\dot\phi}{2X}(2\beta_2+\beta_3)\,, && \text{(B.8)} \\
A_9 &= 2X(a_3+a_4) = \frac{M^2}{4X}(4\beta_2+\beta_3)\,, && \text{(B.9)} \\
B_1 &= 4X\ddot\phi a_5 - 3(H\dot\phi+\ddot\phi)a_3 - 3(H\dot\phi+\ddot\phi)a_{2X} - 2X\ddot\phi a_{3X} - \dot Xa_{2XX} - 4Xa_{3\phi} - \dot\phi a_{2\phi X}\,, \\
& && \text{(B.10)} \\
B_2 &= -2\left[2a_2 + (a_3 - 2a_{2X})X\right]\,, && \text{(B.11)} \\
B_3 &= -2a_2 = -\frac{M^2}{2X}\alpha_T\,, && \text{(B.12)} \\
B_4 &= -4(a_2 - f_X) = -\frac{M^2}{2X}\alpha_H\,, && \text{(B.13)} \\
B_5 &= 2\left[a_2 + (a_3+2a_4)X - f_X\right] = \frac{M^2}{2X}(2\beta_2+\beta_3)\,, && \text{(B.14)} \\
B_6 &= -2\dot\phi(a_3+a_4) = -\frac{M^2\dot\phi}{4X^2}(4\beta_2+\beta_3)\,, && \text{(B.15)} \\
C_1 &= (a_3 + a_{2X})\,, && \text{(B.16)} \\
C_2 &= a_3 + a_4 = \frac{M^2}{8X^2}(4\beta_2+\beta_3)\,. && \text{(B.17)}
\end{aligned}$$

In this appendix the matching between the effective DHOST coefficients and the effective coefficients introduced in [15] is also produced.

Appendix C

Coefficients of the master equation

$$\begin{aligned} T(t) &= -2A_7(t)B_4(t)^2 + 2A_4(t)B_4(t)B_5(t) - 2A_5(t)B_5(t)^2 + 2A_4(t)^2C_2(t) \\ &\quad - 8A_5(t)A_7(t)C_2(t) \end{aligned} \tag{C.1}$$

$$\begin{aligned} R(t) &= 32A_5(t)B_2(t)^2 - 32A_4(t)B_2(t)B_3(t) + 32A_7(t)B_3(t)^2 - 4A_4(t)B_2(t)B_4(t) \\ &\quad + 8A_7(t)B_3(t)B_4(t) + 8A_5(t)B_2(t)B_5(t) - 4A_4(t)B_3(t)B_5(t) + 4A_4(t)^2C_1(t) \\ &\quad - 16A_5(t)A_7(t)C_1(t) \end{aligned} \tag{C.2}$$

$$S_1(t) = -8A_5(t)B_2(t) + 4A_4(t)B_3(t) \tag{C.3}$$

$$S_2(t) = A_4(t)B_4(t) - 2A5(t)B_5(t) \tag{C.4}$$

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